IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Algebraic Bethe ansatz for a spin-½ quantum linear chain with competing interactions

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2000 J. Phys. A: Math. Gen. 33 L439

(http://iopscience.iop.org/0305-4470/33/46/101)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.123
The article was downloaded on 02/06/2010 at 08:35

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **33** (2000) L439–L445. Printed in the UK PII: S0305-4470(00)12792-1

LETTER TO THE EDITOR

Algebraic Bethe ansatz for a spin-$\frac{1}{2}$ quantum linear chain with competing interactions

M A Rego-Monteiro

Centro Brasileiro de Pesquisas Físicas-CBPF/CNPq, Rua Xavier Sigaud, 150 22290-180 Rio de Janeiro, RJ, Brazil

E-mail: regomont@cbpf.br

Received 22 March 2000, in final form 7 August 2000

Abstract. We propose a generalization of the formula that gives an integrable quantum 1D spin Hamiltonian with nearest-neighbour interactions as a logarithmic derivative of a vertex model transfer matrix in order to include in this scheme more realistic integrable models. We compute exactly this generalized formula using the *R* matrix of the *XXX* model, obtaining the Majumdar–Ghosh Hamiltonian plus a charge-like interaction term. We diagonalize this Hamiltonian using the quantum inverse scattering method and present the Bethe ansatz equations of the model.

Since the pioneering work of Bethe in 1931 [1], low-dimensional integrable spin models have been the subject of increasing interest. Nowadays, there is a long list of one-dimensional (1D) integrable spin models solved with the Bethe ansatz or with other methods [2]. Nevertheless, few solutions have been found in higher dimensions and in low-dimensional models involving more complicated interactions such as nearest- and next-nearest-neighbour interactions.

In this Letter we propose a generalization of the well known connection [3] between the transfer matrix of vertex lattice models and quantum 1D spin Hamiltonians with nearest-neighbour interactions without spoiling integrability. The purpose of this extension is to accommodate, within this picture, more realistic integrable quantum 1D spin Hamiltonians presenting nearest-neighbour as well as next-nearest-neighbour interactions.

In our generalization we work with two transfer matrices: one of them is constructed taking the trace of products of L operators with alternating values of the spectral parameter, and the other is given by a shift of the previous one. Apart from a trivial constant, the spin Hamiltonian is given by the difference of the logarithmic derivative of both transfer matrices.

We compute exactly this generalized formula using the R matrix of the *XXX* model, obtaining a spin-$\frac{1}{2}$ quantum 1D Hamiltonian with isotropic nearest- and next-nearest-neighbour interactions (usually called the Majumdar–Ghosh Hamiltonian [4]) plus a charge-like interaction term. Thus, for the R matrix under consideration, our approach provides a model of great interest: a quantum linear chain with competing interactions [5].

Due to the special way this Hamiltonian is given through a transfer matrix, we show that we can diagonalize it using the quantum inverse scattering model (QISM) [6], obtaining the Bethe vectors, energy eigenvalues and the algebraic Bethe ansatz equations (BAE) for this model, generalizing in a non-trivial way the respective well known expressions of the *XXX* Heisenberg model.

0305-4470/00/460439+07$30.00 © 2000 IOP Publishing Ltd

The results of this Letter indicate that, associated with each model in the large class of integrable spin models obtained by means of a transfer matrix of vertex models, there is a descendant integrable model with nearest- and next-nearest-neighbour interactions obtained following the approach we are going to describe.

We start by presenting the main points of a generalization of the QISM that is suitable for the discussion of that Hamiltonian with competing interactions we are going to study. The main formula of the QISM is a variation of the Yang–Baxter relation [7], usually called the RLL relation:

$$R_{a_1a_2}(\lambda-\mu)L_{n,a_1}(\lambda)L_{n,a_2}(\mu) = L_{n,a_2}(\mu)L_{n,a_1}(\lambda)R_{a_1a_2}(\lambda-\mu) \tag{1}$$

where λ is a complex parameter called the spectral parameter and $R_{a_1a_2}(\lambda)$ is the well known R matrix of the *XXX* model [6] that acts in the tensor product of two auxiliary spaces given by $C^2 \otimes C^2$ and satisfies the quantum Yang–Baxter relation [7]. $L_{n_1,a_1}(\lambda)$ is an operator acting in the tensor product of a local space C^2 and an auxiliary C^2:

$$L_{n,a}(\lambda) = \lambda I_n \otimes I_a + \frac{\mathrm{i}}{2}\sum_\alpha \sigma_n^\alpha \otimes \sigma^\alpha \tag{2}$$

with I denoting the unit matrix in the respective spaces and $\vec{\sigma}$ the Pauli matrices.

As noted in [8], the RLL relation can be generalized as

$$R_{a_1a_2}(\lambda-\mu)L_{n,a_1}(\lambda^{(n)})L_{n_1,a_2}(\mu^{(n)}) = L_{n_1,a_2}(\mu^{(n)})L_{n_1,a_1}(\lambda^{(n)})R_{a_1,a_2}(\lambda-\mu) \tag{3}$$

where $\lambda^{(n)} = \lambda + v_n$ and $\mu^{(n)} = \mu + v_n$, with v_n a set of complex numbers. This allow us to define

$$T_a(\{\lambda\}) = L_{1,a}(\lambda^{(1)})\ldots L_{n,a}(\lambda^{(n)})\ldots L_{N,a}(\lambda^{(N)}) \tag{4}$$

which can be written in matrix form as

$$T_a(\{\lambda\}) = \begin{pmatrix} A(\{\lambda\}) & B(\{\lambda\}) \\ C(\{\lambda\}) & D(\{\lambda\}) \end{pmatrix}. \tag{5}$$

It is easy to see that [8]

$$R_{a_1a_2}(\lambda-\mu)T_{a_1}(\{\lambda\})T_{a_2}(\{\mu\}) = T_{a_2}(\{\mu\})T_{a_1}(\{\lambda\})R_{a_1a_2}(\lambda-\mu) \tag{6}$$

which is the generalization of the usual RTT equations.

From the above RTT equations we get

$$\begin{aligned} &[B(\{\lambda\}), B(\{\mu\})] = 0 \\ &A(\{\lambda\})B(\{\mu\}) = f(\lambda-\mu)B(\{\mu\})A(\{\lambda\}) + g(\lambda-\mu)B(\{\lambda\})A(\{\mu\}) \\ &D(\{\lambda\})B(\{\mu\}) = h(\lambda-\mu)B(\{\mu\})D(\{\lambda\}) + k(\lambda-\mu)B(\{\lambda\})D(\{\mu\}) \end{aligned} \tag{7}$$

where

$$f(\lambda) = \frac{\lambda-\mathrm{i}}{\lambda} \qquad g(\lambda) = \frac{\mathrm{i}}{\lambda} \qquad h(\lambda) = \frac{\lambda+\mathrm{i}}{\lambda} \qquad k(\lambda) = \frac{\mathrm{i}}{\lambda}. \tag{8}$$

Consider the reference state $\Omega = \prod_{n=N}^{1} \otimes w_n$ with $w_n \equiv |\uparrow\rangle$. On this state we have

$$L_n(\lambda^{(n)})w_n = \begin{pmatrix} \lambda^{(n)} + \mathrm{i}/2 & * \\ 0 & \lambda^{(n)} - \mathrm{i}/2 \end{pmatrix} w_n \tag{9}$$

with $*$ denoting operator expressions which are not relevant for us. Thus we have

$$C(\{\lambda\})\Omega = 0 \qquad A(\{\lambda\})\Omega = \alpha^N(\{\lambda\})\Omega \qquad D(\{\lambda\})\Omega = \delta^N(\{\lambda\})\Omega \tag{10}$$

with $\alpha^N(\{\lambda\}) \equiv \prod_{i=1}^N \alpha(\lambda^{(\mathrm{i})})$, $\delta^N(\{\lambda\}) \equiv \prod_{i=1}^N \delta(\lambda^{(\mathrm{i})})$, $\alpha(\lambda) = \lambda + \mathrm{i}/2$ and $\delta(\lambda) = \lambda - \mathrm{i}/2$. From the above relations we can easily see that Ω is an eigenstate of $F(\{\lambda\}) = A(\{\lambda\}) + D(\{\lambda\})$.

Now, we define vectors of the form

$$\phi(\{\lambda\}) = B(\{\lambda\}_1)\dots B(\{\lambda\}_l)\Omega. \tag{11}$$

Using equations (7) we can show that

$$[A(\{\lambda\}) + D(\{\lambda\}) - A(\{\tilde{\lambda}\}) - D(\{\tilde{\lambda}\})]\,\phi(\{\lambda\}) = [\Lambda(\{\lambda\}) - \Lambda(\{\tilde{\lambda}\})]\,\phi(\{\lambda\}) \tag{12}$$

where $\tilde{\lambda}^{(n)} = \lambda^{(n)} + \psi$ with ψ complex and

$$\Lambda(\{\lambda\}) = \alpha^N(\{\lambda\})\prod_{k=1}^{\ell} f(\lambda - \lambda_k) + \delta^N(\{\lambda\})\prod_{k=1}^{\ell} h(\lambda - \lambda_k) \tag{13}$$

if each λ_k satisfies

$$\alpha^N(\{\lambda\}_k)\prod_{m\neq k}^{\ell} f(\lambda_k - \lambda_m) = \delta^N(\{\lambda\}_k)\prod_{m\neq k}^{\ell} h(\lambda_k - \lambda_m). \tag{14}$$

Let us now consider the special T matrix given by

$$T(\{\lambda\}) = L_{1,a}(\lambda^{(1)})\dots L_{n,a}(\lambda^{(n)})\dots L_{N,a}(\lambda^{(N)}) \tag{15}$$

where

$$\lambda^{(\mathrm{odd})} = \lambda \qquad \lambda^{(\mathrm{even})} = \lambda + \mathrm{i}\alpha \tag{16}$$

and

$$T(\{\tilde{\lambda}\}) = L_{1,a}(\tilde{\lambda}^{(1)})\dots L_{N,a}(\tilde{\lambda}^{(N)}) \tag{17}$$

for $\tilde{\lambda}^{(n)} = \lambda^{(n)} + \psi$ with $\psi = -\mathrm{i} - \mathrm{i}\alpha$. Defining the transfer matrix F

$$F(\{\lambda\}) = \mathrm{tr}_a\; T(\{\lambda\}) \tag{18}$$

we have computed exactly

$$H = c\frac{\mathrm{d}}{\mathrm{d}\lambda}\{\ln F(\{\lambda\}) - \ln F(\{\tilde{\lambda}\})\}|_{\lambda=\mathrm{i}/2} + N(\alpha-1)(\alpha+2) \tag{19}$$

for $c = -\mathrm{i}(\alpha-1)(\alpha+1)$, using both Maple and Mathematica software (it can be used in any algebraic computation software), for four and six sites and we have obtained

$$H = \sum_{n=1}^{2M}\vec{\sigma}_n\cdot\vec{\sigma}_{n+1} - \frac{\alpha^2}{2}\sum_{n=1}^{2M}\vec{\sigma}_n\cdot\vec{\sigma}_{n+2} + \frac{\mathrm{i}\alpha}{2}\sum_{n=1}^{2M}(-1)^n\varepsilon^{ijk}\sigma_n^i\sigma_{n+1}^j\sigma_{n+2}^k + N\frac{(7\alpha^2+4\alpha-8)}{8} \tag{20}$$

with $\sigma_{n+N}^i = \sigma_n^i$, ε^{ijk} the totally antisymmetric Levi-Civita tensor and $M = N/2$.

In two cases the Hamiltonian in (20) can be computed exactly for arbitrary even $N \geqslant 4$: for $\alpha \gg 1$ and α infinitesimal. In these cases we obtain exactly the asymptotic values of the Hamiltonian in (20) (see the appendix for the computation). As a consequence of the asymptotic analysis, if there is any other term in (20) for $0 < \alpha < \infty$ and arbitrary even $N \geqslant 4$, this extra term would contribute to the matrix element of H for $\alpha \to 0$ (or $\alpha \to \infty$) as $h_{(i,j)}^{\mathrm{extra}} \to \alpha^{r_{(i,j)}}$ for $1 < r_{(i,j)} < 2$. But, since by construction the matrix elements of T are polynomials with integer powers in α, the matrix elements of H as computed from equations (15)–(19) are, in general, fractions of polynomials with integer powers in α. Thus, as α goes to zero or infinity these matrix elements of H behave as $\alpha^{n_{(i,j)}}$, where $n_{(i,j)}$ are integers, implying that the extra terms are zero. Then, the Hamiltonian obtained in equation (20) is the general result for arbitrary even $N \geqslant 4$.

The above Hamiltonian is the periodic Majumdar–Ghosh Hamiltonian plus a $SU(2)$-invariant charge-like interaction term. The Majumdar–Ghosh Hamiltonian is conjectured not to be integrable [9] and what comes out from our calculation is that the additional charge-like interaction term we find in equation (20) is essential to render it integrable.

Let us call the third term of equation (20)

$$Q = \alpha \sum_{n=1}^{2M} (-1)^n \varepsilon^{\mu\nu\rho} \sigma_n^{\mu} \sigma_{n+1}^{\nu} \sigma_{n+2}^{\rho}$$

where Q is Hermitian for $\alpha =$ pure imaginary. For complex α (not pure imaginary) we must make projections on the space of states of the system in order to obtain a model with a Hermitian Hamiltonian. This projection will select from among the eigenstates of the Hamiltonian given by the first two terms in equation (20) for complex α (not pure imaginary) those states belonging to the sector of zero eigenvalue of Q.

The case where $\alpha = \mathrm{i}$ is particularly interesting since the ground state of the first two terms in equation (20) is known exactly and it has a dimerized form [4, 11]. If we introduce the notation for the singlet pair as

$$[l, m] \equiv \frac{1}{\sqrt{2}} (|\uparrow\rangle_l \otimes |\downarrow\rangle_m - |\downarrow\rangle_l \otimes |\uparrow\rangle_m) \tag{21}$$

and define $V_1(N)$ and $V_2(N)$ as

$$\begin{aligned} V_1(N) &\equiv [1, 2][3, 4][5, 6] \ldots [N-1, N] \\ V_2(N) &\equiv [2, 3][4, 5][6, 7] \ldots [N, 1] \end{aligned} \tag{22}$$

we know that $V_{1,2}(N)$ are ground states of the first two terms of equation (20) for $\alpha = \mathrm{i}$. It is possible to verify that there is no $c_{1,2}$ given in

$$V(N) = c_1 V_1(N) + c_2 V_2(N) \tag{23}$$

such that $V(N)$ is an eigenstate of Q. Then, the ground state of the Majumdar–Ghosh model at the Majumdar–Ghosh point (equation (20) with the first two terms for $\alpha = \mathrm{i}$) does not remain the ground state of the Hamiltonian given in equation (20) for $\alpha = \mathrm{i}$.

As this Hamiltonian is derived from the transfer matrix $F(\{\lambda\})$ defined by equations (18) and (19) it can be diagonalized as discussed previously with eigenvectors

$$\phi(\{\lambda\}) = B(\{\lambda\}_1) \ldots B(\{\lambda\}_l)\Omega \tag{24}$$

where λ_k, $k = 1, \ldots, \ell$, satisfy the algebraic Bethe ansatz equations

$$\frac{(\lambda_k + \mathrm{i}/2)^M}{(\lambda_k - \mathrm{i}/2)^M} \frac{(\lambda_k + \mathrm{i}/2 + \mathrm{i}\alpha)^M}{(\lambda_k - \mathrm{i}/2 + \mathrm{i}\alpha)^M} = \prod_{m \neq k}^{\ell} \frac{\lambda_k - \lambda_m + \mathrm{i}}{\lambda_k - \lambda_m - \mathrm{i}} \tag{25}$$

with energy eigenvalue

$$E = (\alpha - 1)(\alpha + 1) \sum_{\beta=1}^{l} \left\{ \frac{1}{\lambda_\beta^2 + \frac{1}{4}} + \frac{1}{\lambda_\beta^2 + 2\mathrm{i}\alpha + (\frac{1}{2} + \alpha)(\frac{1}{2} - \alpha)} \right\}. \tag{26}$$

Of course, if we perform the limit $\alpha \to 0$ then equations (20), (25) and (26) become the Hamiltonian, BAE and energy eigenvalue of the Heisenberg *XXX* quantum chain, respectively.

It is possible to prove that

$$[S^3, B(\{\lambda\})] = -B(\{\lambda\}) \qquad [S^+, B(\{\lambda\})] = A(\{\lambda\}) - D(\{\lambda\}). \tag{27}$$

Since for the reference state Ω we have

$$S^+\Omega = 0 \qquad S^3\Omega = \frac{N}{2}\Omega \tag{28}$$

using equations (27) and repeating the procedure that was used to derive the BAE [12] we can show that if the BAE, equation (25), are satisfied we have

$$S^{+}\phi(\{\lambda\}) = 0 \tag{29}$$

which means that $\phi(\{\lambda\})$ are all highest weight states.

The connection between the transfer matrix of vertex lattice models and quantum 1D spin Hamiltonians with nearest-neighbour interactions is well known. Several integrable quantum 1D spin models with nearest-neighbour interactions are within this framework. We believe that the case analysed in this Letter (acquisition of the integrable model given in equation (20) having nearest-neighbour as well as next-nearest-neighbour interactions through a vertex transfer matrix and the first steps in the proof of complete integrability of the model) is not a singular case. We conjecture that, associated with each model in the large class of integrable spin models with nearest-neighbour interactions obtained by means of a transfer matrix of vertex models, there is a descendant integrable spin Hamiltonian with nearest-neighbour as well as next-nearest-neighbour interactions obtained using the approach described in this Letter.

Finally, it would be interesting to investigate if equations (18) and (19) could be further generalized in order to accommodate integrable descendants with interactions up to nth-neighbour interactions.

The author thanks L Rodrigues for discussions in the early stages of this work, S Sciuto and a member of the Editorial Board for useful critical comments on the manuscript and PRONEX/FINEP/MCT for partial support.

Appendix

In this appendix we are going to prove that asymptotic limits of the Hamiltonian in (20) are obtained using equations (15)–(19) in two asymptotic cases. Details are given just for the λ part of the transfer matrix in equation (19), since the $\tilde{\lambda}$ part is obtained in a similar way. Moreover, for simplicity we will focus our analysis on the terms proportional to α.

It is convenient to rewrite the L matrices for odd and even sites as

$$L_{n,a}(\lambda) = (\lambda - \mathrm{i}/2)I_{n,a} + \mathrm{i}P_{n,a} \tag{30}$$

and

$$\hat{L}_{n,a}(\lambda) = (\lambda - \mathrm{i}/2)I_{n,a} + \mathrm{i}P_{n,a} + \mathrm{i}\alpha I_{n,a} \tag{31}$$

where we denote by $L_{n,a_1}(\lambda)$ the L matrix for odd sites, $\hat{L}_{n,a_1}(\lambda)$ the L matrix for even sites and $P_{n,a}$ is the twist matrix for quantum and auxiliary spaces indicated by the sub-indices n and a, respectively. Moreover, in components these matrices are written as

$$L_{n,a}(\lambda) \longrightarrow L_{\alpha_n}^{\bar{\alpha}_n}(\gamma_n\gamma_{n+1})(\lambda) \tag{32}$$

where $(\alpha_n, \bar{\alpha}_n)$ are the quantum indices and $(\gamma_n\gamma_{n+1})$ the auxiliary indices. Using these notations we write the transfer matrix as

$$F_{\{\alpha\}}^{\{\bar{\alpha}\}}(\{\lambda\}) = \sum_{\{\gamma\}} L_{\alpha_1}^{\bar{\alpha}_1}(\gamma_1\gamma_2)(\lambda)\hat{L}_{\alpha_2}^{\bar{\alpha}_2}(\gamma_2\gamma_3)(\lambda)\ldots L_{\alpha_{N-1}}^{\bar{\alpha}_{N-1}}(\gamma_{N-1}\gamma_N)(\lambda)L_{\alpha_N}^{\bar{\alpha}_N}(\gamma_N\gamma_1)(\lambda). \tag{33}$$

In two cases the Hamiltonian in (20) can be computed exactly for arbitrary even $N \geqslant 4$. In the first case we take $\alpha \gg 1$ in equations (15)–(19) and, since in equation (31) the dominant term is unity in the auxiliary and quantum spaces, we trivially obtain, apart from a trivial constant that can be easily computed, the isotropic interaction spanning over odd sites. It can be easily verified that the $\tilde{\lambda}$ part of the transfer matrix in equation (19) gives the isotropic interaction spanning over even sites.

The second case, obtained by considering α infinitesimal in equations (15)–(19), is less trivial. Consider the transfer matrix when $\lambda = \mathrm{i}/2$ for α infinitesimal:

$$F(\{\mathrm{i}/2\}) = A + \alpha \sum_{i=1}^{N/2-1} B(i) \tag{34}$$

where

$$A^{\{\bar{\alpha}\}}_{\{\alpha\}} = \mathrm{i}^N \sum_{\{\gamma\}} P^{\bar{\alpha}_1}_{\alpha_1}(\gamma_1\gamma_2)\dots P^{\bar{\alpha}_N}_{\alpha_N}(\gamma_N\gamma_1) \tag{35}$$

and

$$B^{\{\bar{\alpha}\}}_{\{\alpha\}}(i) = \mathrm{i}^N \sum_{\{\gamma\}} P^{\bar{\alpha}_1}_{\alpha_1}(\gamma_1\gamma_2)\dots P^{\bar{\alpha}_{2i-1}}_{\alpha_{2i-1}}(\gamma_{2i-1}\gamma_{2i}) I^{\bar{\alpha}_{2i}}_{\alpha_{2i}}(\gamma_{2i}\gamma_{2i+1}) P^{\bar{\alpha}_{2i+1}}_{\alpha_{2i+1}}(\gamma_{2i+1}\gamma_{2i+2})\dots P^{\bar{\alpha}_N}_{\alpha_N}(\gamma_N\gamma_1). \tag{36}$$

It is easy to see that the inverse transfer matrix for $\lambda = \mathrm{i}/2$ in the α infinitesimal case is

$$F^{-1}(\{\mathrm{i}/2\}) = A - \alpha \sum_{i=1}^{N/2} B(i). \tag{37}$$

Moreover

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} F(\{\lambda\})|_{\lambda=\mathrm{i}/2} = \sum_{n=1}^{N} C(n) + \alpha\left(\sum_{i=0}^{N/2-1} D(i) + \sum_{i=1}^{N/2} E(i)\right) \tag{38}$$

where,

$$C^{\{\bar{\alpha}\}}_{\{\alpha\}}(n) = \mathrm{i}^{N-1} \sum_{\{\gamma\}} P^{\bar{\alpha}_1}_{\alpha_1}(\gamma_1\gamma_2)\dots P^{\bar{\alpha}_{n-1}}_{\alpha_{n-1}}(\gamma_{n-1}\gamma_n) \dot{L}^{\bar{\alpha}_n}_{\alpha_n}(\gamma_n\gamma_{n+1}) P^{\bar{\alpha}_{n+1}}_{\alpha_{n+1}}(\gamma_{n+1}\gamma_{n+2})\dots P^{\bar{\alpha}_N}_{\alpha_N}(\gamma_N\gamma_1) \tag{39}$$

$$\begin{aligned} D^{\{\bar{\alpha}\}}_{\{\alpha\}}(i) = \mathrm{i}^{N-1} \sum_{n=1}^{N/2} \sum_{\{\gamma\}} & P^{\bar{\alpha}_1}_{\alpha_1}(\gamma_1\gamma_2)\dots P^{\bar{\alpha}_{2i}}_{\alpha_{2i}}(\gamma_{2i}\gamma_{2i+1}) \dot{L}^{\bar{\alpha}_{2i+1}}_{\alpha_{2i+1}}(\gamma_{2i+1}\gamma_{2i+2}) P^{\bar{\alpha}_{2i+2}}_{\alpha_{2i+2}}(\gamma_{2i+2}\gamma_{2i+3}) \\ & \dots P^{\bar{\alpha}_{2n-1}}_{\alpha_{2n-1}}(\gamma_{2n-1}\gamma_{2n}) I^{\bar{\alpha}_{2n}}_{\alpha_{2n}}(\gamma_{2n}\gamma_{2n+1}) P^{\bar{\alpha}_{2n+1}}_{\alpha_{2n+1}}(\gamma_{2n+1}\gamma_{2n+2})\dots P^{\bar{\alpha}_N}_{\alpha_N}(\gamma_N\gamma_1) \end{aligned} \tag{40}$$

and

$$\begin{aligned} E^{\{\bar{\alpha}\}}_{\{\alpha\}}(i) = \mathrm{i}^{N-1} \sum_{n=1,n\neq i}^{N/2} \sum_{\{\gamma\}} & P^{\bar{\alpha}_1}_{\alpha_1}(\gamma_1\gamma_2)\dots P^{\bar{\alpha}_{2i-1}}_{\alpha_{2i-1}}(\gamma_{2i-1}\gamma_{2i}) \dot{L}^{\bar{\alpha}_{2i}}_{\alpha_{2i}}(\gamma_{2i}\gamma_{2i+1}) P^{\bar{\alpha}_{2i+1}}_{\alpha_{2i+1}}(\gamma_{2i+1}\gamma_{2i+2}) \\ & \dots P^{\bar{\alpha}_{2n-1}}_{\alpha_{2n-1}}(\gamma_{2n-1}\gamma_{2n}) I^{\bar{\alpha}_{2n}}_{\alpha_{2n}}(\gamma_{2n}\gamma_{2n+1}) P^{\bar{\alpha}_{2n+1}}_{\alpha_{2n+1}}(\gamma_{2n+1}\gamma_{2n+2})\dots P^{\bar{\alpha}_N}_{\alpha_N}(\gamma_N\gamma_1) \end{aligned} \tag{41}$$

with

$$\dot{L}^{\bar{\alpha}_m}_{\alpha_m}(\gamma_m\gamma_{m+1}) = \frac{\mathrm{d}}{\mathrm{d}\lambda} L^{\bar{\alpha}_m}_{\alpha_m}(\gamma_m\gamma_{m+1})(\lambda)|_{\lambda=\mathrm{i}/2}. \tag{42}$$

Now, we are going to compute the product $F^{-1}(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}F(\{\lambda\})|_{\lambda=\mathrm{i}/2}$ from equations (37) and (38). The product of the first terms of the right-hand side of equations (37) and (38) is a well known calculation and gives the first term of the right-hand side of equation (20). It can be verified that the first term on the right-hand side of equation (37) times the second term on the right-hand side of equation (38) plus the second term on the right-hand side of equation (37) times the terms with n odd in the first term on the right-hand side of equation (38) gives

$$\mathrm{i}^{2N-1} \sum_{i=0}^{N/2-1} \delta^{\bar{\alpha}_1}_{\alpha_1}\dots\delta^{\bar{\alpha}_{2i}}_{\alpha_{2i}}(\delta^{\bar{\alpha}_{2i+1}}_{\alpha_{2i+2}} \dot{L}^{\bar{\alpha}_{2i+2}}_{\alpha_{2i+3}}(\alpha_{2i+1}\bar{\alpha}_{2i+3}) - \delta^{\bar{\alpha}_{2i+2}}_{\alpha_{2i+1}} \dot{L}^{\bar{\alpha}_{2i+1}}_{\alpha_{2i+3}}(\alpha_{2i+2}\bar{\alpha}_{2i+3}))\delta^{\bar{\alpha}_{2i+4}}_{\alpha_{2i+4}}\dots\delta^{\bar{\alpha}_N}_{\alpha_N}. \tag{43}$$

Moreover, the first term on the right-hand side of equation (37) times the third term on the right-hand side of equation (38) plus the second term on the right-hand side of equation (37) times the n-even terms in the first term on the right-hand side of equation (38) gives

$$-\mathrm{i}^{2N-1}\sum_{i=1}^{N/2}\delta_{\alpha_1}^{\bar{\alpha}_1}\ldots\delta_{\alpha_{2i-2}}^{\bar{\alpha}_{2i-2}}\dot{L}_{\alpha_{2i}}^{\bar{\alpha}_{2i}}(\alpha_{2i-1}\bar{\alpha}_{2i-1})\delta_{\alpha_{2i+1}}^{\bar{\alpha}_{2i+1}}\ldots\delta_{\alpha_N}^{\bar{\alpha}_N}. \qquad (44)$$

It is easy to see that

$$\delta_{\alpha_{2i+2}}^{\bar{\alpha}_{2i+1}}\dot{L}_{\alpha_{2i+3}}^{\bar{\alpha}_{2i+2}}(\alpha_{2i+1}\bar{\alpha}_{2i+3})-\delta_{\alpha_{2i+1}}^{\bar{\alpha}_{2i+2}}\dot{L}_{\alpha_{2i+3}}^{\bar{\alpha}_{2i+1}}(\alpha_{2i+2}\bar{\alpha}_{2i+3})=-\frac{\mathrm{i}}{2}\varepsilon^{lmn}\sigma^{l}_{\alpha_{2i+1}\bar{\alpha}_{2i+1}}\sigma^{m}_{\alpha_{2i+2}\bar{\alpha}_{2i+2}}\sigma^{n}_{\alpha_{2i+3}\bar{\alpha}_{2i+3}} \qquad (45)$$

and

$$\dot{L}_{\alpha_{2i}}^{\bar{\alpha}_{2i}}(\alpha_{2i-1}\bar{\alpha}_{2i-1})=\delta_{\alpha_{2i-1}}^{\bar{\alpha}_{2i-1}}\delta_{\alpha_{2i}}^{\bar{\alpha}_{2i}}. \qquad (46)$$

Using equations (39)–(42) in the λ part of the transfer matrix in equation (19) we obtain the odd sites of the third term and half of the term proportional to α in the fourth term of equation (20). The even sites are obtained by a similar computation using the $\tilde{\lambda}$ part of the transfer matrix in equation (19).

References

[1] Bethe H 1931 *Z. Phys.* **71** 205
[2] Baxter R J 1989 *Adv. Stud. Pure Math.* **19** 95
[3] Lieb E H 1967 *Phys. Rev. Lett.* **18** 1046
Sutherland B 1970 *J. Math. Phys.* **11** 3183
Baxter R J 1971 *Phys. Rev. Lett.* **26** 834
[4] Majumdar C K and Ghosh D K 1969 *J. Math. Phys.* **10** 1398
[5] See, for example, Diep H T (ed) 1994 *Magnetic Systems with Competing Interactions* (Singapore: World Scientific)
[6] Faddeev L D, Sklyanin E K and Takhtajan L A 1979 *Theor. Math. Phys.* **40** 194
Faddeev L D 1982 *Les Houches Session XXXIX* (Amsterdam: Elsevier) p 563
[7] McGuire J B 1964 *J. Math. Phys.* **5** 622
Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
[8] De Vega H J 1989 *Int. J. Mod. Phys.* A **4** 2371
[9] Grabowski M P and Mathieu P 1995 *J. Phys. A: Math. Gen.* **28** 4777
[10] See, for instance, Farnell D J J and Parkinson J B 1994 *J. Phys.: Condens. Matter* **6** 5521 and references therein
[11] van den Broek P M 1980 *Phys. Lett.* A **77** 261
Shastry B S and Sutherland B 1981 *Phys. Rev. Lett.* **47** 964
[12] Takhtajan L A 1985 *Lecture Notes in Physics* vol 242, ed B S Shastry, S S Jha and V Singh (Berlin: Springer) p 175